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$1/f^{\alpha}$ noise from correlations between avalanches in self-organized criticality

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We show that large, slowly driven systems can evolve to a self-organized critical state where long-range temporal correlations between bursts or avalanches produce low-frequency $1/f^{\alpha}$ noise. The avalanches can occur instantaneously in the external time scale of the slow drive, and their event statistics are described by power-law distributions. A specific example of this behavior is provided by numerical simulations of a deterministic "sandpile" model, where a scaling relation links α with the avalanche power-law exponent.

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The ubiquity of $1/f^{\alpha}$ noise in nature is one of the oldest problems in contemporary physics still lacking a generally accepted explanation, despite much effort. The phenomenon is characterized by a $1/f^{\alpha}$ decay with a nontrivial α found in the power spectrum $S(f)$ of a given time signal at low frequencies f; i.e., $\alpha \neq 2n$ for all integers n. In many cases, the time signal describes a transport process. Then $1/f^{\alpha}$ noise is an indication of anomalous behavior compared to conventional transport, as for instance, equilibrium diffusion. Flicker noise has been observed in a huge number of diverse systems, many of which are far from equilibrium and show a bursty, avalanche dynamics. Examples include earthquakes [1,2], combustion fronts [3], chemical reactions [4], flux motion in superconductors [5], and Barkhausen noise [6], to name only a few.

In the search for a general dynamical mechanism for $1/f^{\alpha}$ noise in processes far from equilibrium, Bak, Tang, and Wiesenfeld proposed the concept of self-organized criticality (SOC) [7,8]. This refers to the tendency of spatially extended, slowly driven systems to organize into a state with fractal, spatial, and temporal properties that is also characterized by self-similar distributions of event (avalanche) sizes. Although the original "sandpile" model, which was introduced to exemplify the SOC concept, exhibits a scale-free distribution of avalanches, its noise spectrum is of the form $1/f^2$ [9]. Variants of the original sandpile model do indeed exhibit nontrivial $1/f^{\alpha}$ noise, but in these cases [10] the avalanche event distributions are not critical, implying, possibly, mutual exclusivity between a SOC mechanism and mechanisms giving long-range temporal correlations such as $1/f^{\alpha}$ noise. In fact, it has recently been argued in the context of solar flares, transport dynamics in magnetic confinement devices, and other areas, that the presence of temporal correlations between events excludes SOC as an underlying mechanism [11,12]. The argument entails a narrowing of SOC to the phenomenology of certain, specific "sandpile" models, which is in sharp contrast to the original idea [7,8], and is erroneous as we show here.

In order to demonstrate explicitly that SOC can provide a dynamical mechanism which gives correlations between bursts leading to $1/f^{\alpha}$ noise, we study a slowly driven, deterministic sandpile model. It exhibits a power-law distribution of avalanches, as well as $1/f^{\alpha}$ fluctuations in the pattern of dissipation over the slow temporal domain of the external drive. Despite the fact that the time scales of the driving and of the avalanche events are completely separated, the critical behavior of the power spectrum is solely determined by the critical properties of the avalanche size distribution. They are linked by a scaling relation. In addition, the results we find are robust with respect to changes in the definition of the time scale associated with the driving. These observations constitute a proof that a SOC mechanism can give low-frequency $1/f^{\alpha}$ noise due to correlations between power-law distributed avalanches, without imposing temporal correlations in the external drive.

Actually several different models of SOC, describing e.g., traffic [13] and evolution [8,14] do exhibit nontrivial $1/f^{\alpha}$ noise over the temporal domain of individual avalanches. In this context, it is important to distinguish between temporal correlations within the time span of individual avalanche events, and low-frequency noise observed over a much longer temporal regime of an arbitrarily slow external drive. Measured in this external time scale, the individual avalanches can occur almost instantaneously. This is a situation often encountered in Nature. In this case $1/f^{\alpha}$ noise must arise from correlations between avalanches. These correlations can either be induced by long-range temporal correlations in the external drive, or be an intrinsic part of the self-organization process itself. Sanchez *et al.* have found long-range temporal correlations in SOC when the sandpile is driven by an external source of low-frequency noise [12].

However, it is important to clarify if SOC itself can spontaneously generate both critical avalanche statistics and long-range temporal correlations between avalanches, in the presence of a temporally uniform, slow external drive. Surprisingly, this question has not yet been decided, despite the fact that, as in the case of earthquakes, a scale-free avalanche event dynamics (e.g., the Gutenberg-Richter law) can be observed in many far from equilibrium phenomena which exhibit $1/f^{\alpha}$ noise (and non-Poissonian interevent statistics [1,15]) at vastly longer time scales (e.g., days to years) compared to the individual events (e.g., seconds).

The "sandpile" model we discuss was introduced by de Sousa Vieira to describe avalanches in stick-slip phenomena

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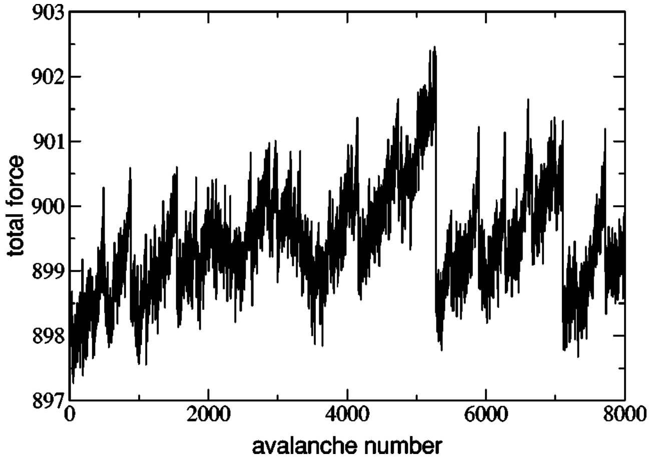

FIG. 1. Fluctuations of the total force in the self-organized critical state for a system of size $L=1024$.

[16]. It is close to the original array of connected pendula first discussed by Bak, Tang, and Wiesenfeld. In detail, the model is defined as follows: Consider a one-dimensional system of size L where a continuous variable $f_n \geqslant 0$ is associated with each site n (representing the force on that site). Initially, all f_n have the same value which is below a threshold f_{th}. The basic time step of the driving force consists in changing the value of the force on the first site according to $f_1 = f_{th} + \delta f$ with a fixed δf. This can be considered as a slow external driving and leads to a fast relaxation process (avalanche) within the system. This relaxation consists of a conservative redistribution of the force at sites with $f_n \geqslant f_{th}$ (toppling sites) according to $f_n = \Phi(f_n - f_{th})$ and $f_{n\pm1} = f_{n\pm1} + \Delta f_n/2$. Here, Δf_n is the change of force at the overcritical site and Φ a periodic, nonlinear function. The relaxation continues until all sites are stable again, i.e., $f_n < f_{th}$ for all n. Then, the driving at the first site sets in again. This definition is complemented by open boundary conditions, i.e., force is lost at both boundaries. Without loss of generality, we use a sequential update and set $f_{th}=1$, $\delta f = 0.1$, $\Phi(x) = 1 - a[x]$ where $[x]$ denotes x modulo $1/a$, i.e., a sawtooth function, and $a=4$. It was shown that the model evolves into a state of SOC where the avalanche distributions are scale free, limited only by the overall system size [16]. Note that like other sandpile models, the toppling rules are conservative, and the total amount of force in the system can only be changed at the boundaries.

An appropriate choice of a time signal to detect $1/f^{\alpha}$ noise, which takes the time scale separation between the slow external driving and the individual avalanches into account, is the total force in the system after each avalanche

$$X(i) = \sum_{n=1}^{L} f_n(i), \tag{1}$$

where i is the avalanche number. This signal is shown in Fig. 1 and directly reveals the stick-slip character of the dynamics.

Analyzing the power spectrum of $X(i)$, we find a clear $1/f^{\alpha}$ decay with a cutoff at low frequencies that shifts to even lower frequencies with increasing system size (see Fig. 2). In particular, a data collapse, shown in the inset, reveals the following scaling behavior:

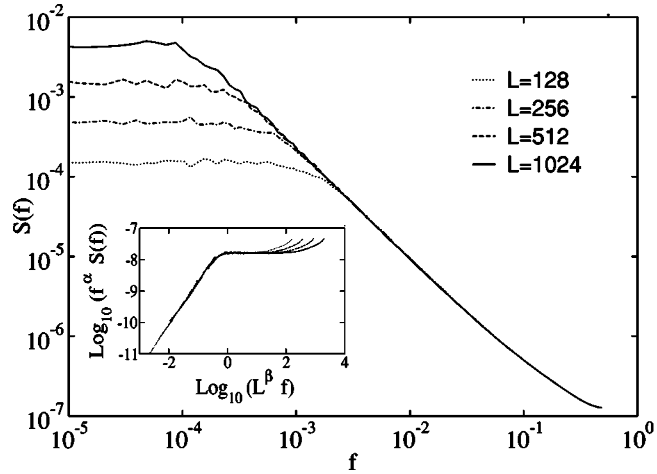

FIG. 2. Power spectrum of the fluctuations in the total force for different system sizes. Note that the cutoff moves to lower frequencies for larger system sizes. Inset: Data collapse with $\alpha=1.38$ and $\beta=1.2$.

$$S(f) \sim \frac{1}{f^{\alpha}} g\left(\frac{1}{fL^{\beta}}\right), \tag{2}$$

with $\alpha=1.38$ and $\beta=1.2$. Here, g is a scaling function that is constant for small arguments and decays as $x^{-\alpha}$ for large arguments. These results are somewhat dependent on the details of the definition of the model [17]. This does not change our main conclusion regarding SOC as a possible mechanism for $1/f^{\alpha}$ noise, although a more robust and general model would be desirable.

To understand how the long-range temporal correlations arise in this model, we have analyzed the fluctuations in the force at a single site, i.e., $X_n(i) = f_n(i)$. The local power spectra are simple Lorentzians with a characteristic frequency that decreases with increasing distance from the driving site. Thus, different time signals of the same system, such as the local and the total force signal, behave in a totally different manner. This has been found for a variety of other noises due to transport of conserved quantities [18]. Although the presence of many Lorentzian contributions with different characteristic time scales t_c is reminiscent of many systems in which $1/f$ noise comes from the distribution of t_c's [18], in this case the overall spectrum is dominated not by the sum of the local spectra from independent different sites but rather by the cross spectra between the linked sites. The sum of the local spectra and the spectrum of the sum have very different α's here [19]. Consequently, the long-range temporal correlations, implied by the occurrence of $1/f^{\alpha}$ noise, are stored in the spatial correlations embedded in the whole system.

This is further confirmed by the fact that scaling relations connect α and β to the critical exponent characterizing the avalanche distributions. The quantity L^{β} in Eq. (2) describes the scaling of the temporal cutoff and, hence, the scaling of the number of avalanches before events become uncorrelated. Thus, it can be related to the avalanche distribution. The distribution of avalanche sizes (the number of toppling events in the avalanche) is distributed as $P(s) \sim s^{-\tau}G(s/L^{D})$, where $\tau \simeq 1.54$ is the so-called histogram exponent, and the avalanche dimension $D \simeq 2.20$ gives the cutoff for the largest avalanche in a system of size L [16]. Avalanches in the power-law regime do not extend through the entire system; therefore some sites do not topple and the system retains memory of its previous force. However, avalanches larger than the cutoff, which are those that entirely span the chain, decorrelate the system because all sites topple and get a new force. The frequency of the large system wide events that decorrelate the force in the system scales as $L^{-D(\tau-1)}$. Thus, $\beta = D(\tau-1)$. Also, a scaling relation $\langle s \rangle \sim L$ was found numerically in Ref. [16], and is a general result for many boundary driven SOC systems [20], giving the result that $D(2-\tau)=1$. Combining the two equations, $\beta = D-1 = (\tau-1)/(2-\tau)$, which agrees very well with numerical results.

Using scaling relations together with some previously obtained results concerning universality classes, we can relate the exponent α to the exponent τ or D via the variance in the total force in the system

$$\int S(f)df = \sigma^2 = \left\langle \left(\sum_{n=1}^{L} f_n(i) - \left\langle \sum_{n=1}^{L} f_n(i) \right\rangle \right)^2 \right\rangle, \quad (3)$$

where the integral is over all frequencies. Since the exponent $\alpha > 1$, the power in the signal diverges as $L \to \infty$ as $L^{\beta(\alpha-1)}$. As argued and supported numerically in Ref. [16], the sandpile model is in the same universality class as the original Burridge-Knopoff train model. This latter model has been conjectured to be in the universality class of interface depinning with the interface pulled at one end [20]—a result which is also supported by numerical simulations. Thus, the fluctuations in the force in the system have the dimension of fluctuations of force in the interface depinning problem, e.g., $[\int_0^L dx \nabla^2 H(x)]^2$ (see Ref. [20] for more explanation). Since in that case, the height of the interface H has the dimension of L^{D-1}, then $\sigma^2 \sim L^{2(D-2)}$. Using the previous relation for β, we get $\alpha = (3D-5)/(D-1) = (5\tau-7)/(\tau-1)$. This again agrees very well with our numerical simulation results and is consistent with the results in Ref. [21]. The existence of these scaling relations makes it clear without ambiguity that, in this model, the critical avalanche dynamics and the long-range temporal correlations belong inseparably together.

Instead of using the avalanche number, we can choose different definitions for the slow time scale of the model. Since the force on the first site is set to $f_{th} + \delta f$ to induce a new avalanche, one can choose to identify the amount of force added to the first site with the temporal interval between avalanches. This corresponds to a uniform driving. Such a choice leads to nonuniform time intervals between successive relaxation events which could in principle alter the behavior. The time signal defined in Eq. (1) is modified and becomes a sawtooth function

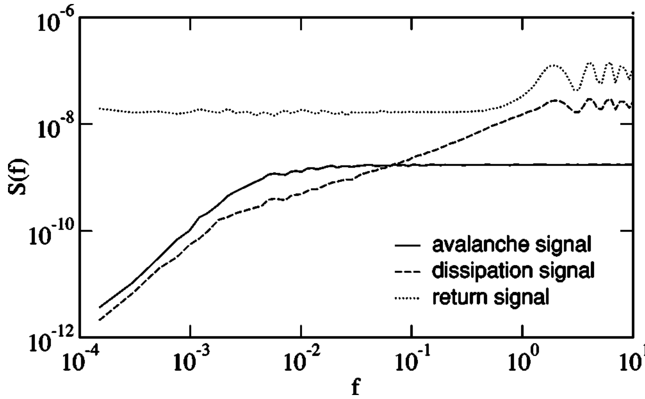

FIG. 3. Power spectrum of the different, differentiated, time signals for $L=256$ (see text). Note that in the avalanche and dissipation signal, an increase with f^2 as observed for the lowest frequencies corresponds to white noise in the integrated signal. The avalanche signal is shifted down by 8 decades.

$$X(t) = \sum_{n=1}^{L} f_n(t) = t - t_i + \sum_{n=1}^{L} f_n(t_i), \quad (4)$$

where $t_i \leq t$ denotes the time of occurrence of the last avalanche. The power spectrum of this time signal shows exactly the same $1/f^{\alpha}$ decay as the signal defined in Eq. (1) as can be deduced from Fig. 3. There, we show the power spectrum of the differentiated signal $X'(t)$—called a dissipation signal for obvious reasons—which follows a power law with $S(f) \propto f^{2-\alpha}$. In particular, the cutoff to white noise behavior $(\alpha=0)$ for very low-frequencies occurs almost at the same frequency as before and the scaling with L is also unchanged. Note that white noise, $[S(f) \sim f^0]$ in $X(t)$ in Fig. 2, corresponds to $S(f) \propto f^2$ in $X'(t)$ in Fig. 3.

The dissipation signal resembles the form of a pulse train, i.e.,

$$X'(t) = \sum_i h_i \delta(t-t_i), \quad (5)$$

where h_i is the dissipation in the total force due to the ith avalanche. This form of the signal is especially well suited to clarify the source of the long time correlations. For instance, substituting h_i by a constant should reveal the correlations induced solely by the fluctuations in the time intervals between dissipation events. As shown in Refs. [1,22], correlations between subsequent time intervals can, indeed, lead to $1/f^{\alpha}$ noise. However, here we find that this "return signal" is uncorrelated (see Fig. 3). The absence of correlations in the time intervals together with the existence of a finite second moment of their distribution due to the boundedness of the intervals by $f_{th} + \delta f$ implies that the power spectrum is the same as before for frequencies below a certain, fixed f_h. Hence, the long time scales are not affected by the fluctuations in the time intervals. In particular, this is true for any distribution of time intervals with finite second moment if the time intervals are independent of one another (central limit theorem). Thus, the $1/f^{\alpha}$ noise is due solely to the fluctuations in h_i. We conclude that the long-range temporal correlations in the model are exclusively encoded in the state of the whole system and are not destroyed by the fluctuations due to changes in the definition of the time scale of the external driving.

The fact that the occurrence of $1/f^\alpha$ noise in the model cannot be attributed to a low-dimensional dynamics shows that the SOC mechanism is totally different and can be well distinguished from other deterministic mechanisms which attempt to explain flicker noise as a chaotic phenomenon (see, e.g., Ref. [23]). Unlike those mechanisms, the dynamics discussed here cannot be reduced to a renewal process with a power-law distribution of waiting times or step sizes.

Finally, the temporal variations in the sizes of the avalanches show only trivial long time correlations. Fig. 3 shows the power spectrum of the avalanche signal, defined in Eq. (5) where h_i is now the number of topplings in the ith avalanche. Clearly, a flat spectrum can be observed with a change to f^2 behavior at low frequencies due to the finite system size. In terms of the integrated signal $X(t)$, this corresponds to a trivial $1/f^2$ decay crossing over to white noise (f^0) behavior at low frequencies. Hence, the underlying process can be considered as a random walk confined by the system size.

The mechanism for flicker noise found here is in contrast to the (almost) equilibrium mechanisms thought to be responsible for $1/f$ noise in, for example, electrical current measurements in solids [18]. Those mechanisms are clearly not applicable to far from equilibrium, avalanching systems. Our results explicitly demonstrate that a SOC mechanism, with a power-law distribution of avalanches, can lead to long time correlations between avalanches giving low-frequency $1/f^\alpha$ noise in slowly driven, far from equilibrium systems.

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